Engineering Maths First Aid Kit

7.4

The polar form

Introduction.

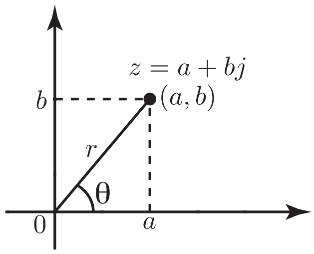
From an Argand diagram the **modulus** and the **argument** of a complex number, can be defined. These provide an alternative way of describing complex numbers, known as the **polar form**. This leaflet explains how to find the modulus and argument.

1. The modulus and argument of a complex number.

The Argand diagram below shows the complex number $z = a + bj$. The distance of the point (a, b) from the origin is called the **modulus**, or **magnitude** of the complex number and has the symbol r. Alternatively, r is written as $|z|$. The modulus is never negative. The modulus can be found using Pythagoras' theorem, that is

$$|z| = r = \sqrt{a^2 + b^2}$$

The angle between the positive x axis and a line joining (a, b) to the origin is called the **argument** of the complex number. It is abbreviated to $\arg(z)$ and has been given the symbol θ.



We usually measure θ so that it lies between $-\pi$ and π, (that is between $-180°$ and $180°$). Angles measured anticlockwise from the positive x axis are conventionally positive, whereas angles measured clockwise are negative. Knowing values for a and b, trigonometry can be used to determine θ. Specifically,

$$\tan\theta = \frac{b}{a} \qquad \text{so that} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

but care must be taken when using a calculator to find an inverse tangent that the solution obtained is in the correct quadrant. Drawing an Argand diagram will always help to identify the correct quadrant. The position of a complex number is uniquely determined by giving its modulus and argument. This description is known as the **polar form**. When the modulus and argument of a complex number, z, are known we write the complex number as $z = r\angle\theta$.

Polar form of a complex number with modulus r and argument θ:

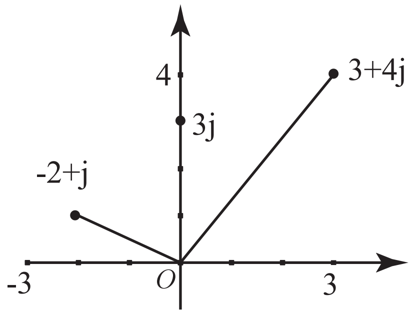
$$z = r\angle\theta$$

Example
Plot the following complex numbers on an Argand diagram and find their moduli.

a) $z_1 = 3 + 4j$, b) $z_2 = -2 + j$, c) $z_3 = 3j$

Solution
The complex numbers are shown in the figure below. In each case we can use Pythagoras' theorem to find the modulus.

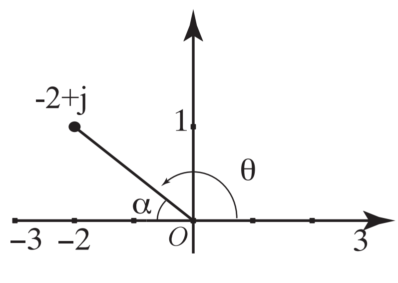
a) $|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, b) $|z_2| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ or 2.236, c) $|z_3| = \sqrt{3^2 + 0^2} = 3$.



Example
Find the arguments of the complex numbers in the previous example.

Solution
a) $z_1 = 3 + 4j$ is in the first quadrant. Its argument is given by $\theta = \tan^{-1}\frac{4}{3}$. Using a calculator we find $\theta = 0.927$ radians, or 53.13°.

b) $z_2 = -2 + j$ is in the second quadrant. To find its argument we seek an angle, θ, in the second quadrant such that $\tan\theta = \frac{1}{-2}$. To calculate this correctly it may help to refer to the figure below in which α is an acute angle with $\tan\alpha = \frac{1}{2}$. From a calculator $\alpha = 0.464$ and so $\theta = \pi - 0.464 = 2.678$ radians. In degrees, $\alpha = 26.57°$ so that $\theta = 180° - 26.57° = 153.43°$.



c) $z_3 = 3j$ is purely imaginary. Its argument is $\frac{\pi}{2}$, or 90°.

Exercises
1. Plot the following complex numbers on an Argand diagram and find their moduli and arguments.

a) $z = 9$, b) $z = -5$, c) $z = 1 + 2j$, d) $z = -1 - j$, e) $z = 8j$, f) $-5j$.

Answers
1. a) $|z| = 9$, arg(z)=0, b) $|z| = 5$, $\arg(z) = \pi$, or 180°, c) $|z| = \sqrt{5}$, $\arg(z) = 1.107$ or 63.43°, d) $|z| = \sqrt{2}$, $\arg(z) = -\frac{3\pi}{4}$ or −135°, e) $|z| = 8$, $\arg(z) = \frac{\pi}{2}$ or 90°, f) $|z| = 5$, $\arg(z) = -\frac{\pi}{2}$ or −90°.

