Engineering Maths First Aid Kit

3.8

Solving equations involving logarithms and exponentials

Introduction

It is often necessary to solve an equation in which the unknown occurs as a power, or exponent. For example, you may need to find the value of x which satisfies $2^x = 32$. Very often the base will be the exponential constant e, as in the equation $\mathrm{e}^x = 20$. To understand what follows you must be familiar with the exponential constant. See leaflet *3.4 The exponential constant* if necessary.

You will also come across equations involving logarithms. For example you may need to find the value of x which satisfies $\log_{10} x = 34$. You will need to understand what is meant by a logarithm, and the laws of logarithms (leaflets *2.19 What is a logarithm ?* and *2.20 The laws of logarithms*). On this leaflet we explain how such equations can be solved.

1. Revision of logarithms

Logarithms provide an alternative way of writing expressions involving powers. If

$$a = b^c \qquad \text{then} \qquad \log_b a = c$$

For example: $\quad 100 = 10^2$ can be written as $\log_{10} 100 = 2$.

Similarly, $\mathrm{e}^3 = 20.086$ can be written as $\log_{\mathrm{e}} 20.086 = 3$.

The third law of logarithms states that, for logarithms of any base,

$$\log A^n = n \log A$$

For example, we can write $\log_{10} 5^2$ as $2 \log_{10} 5$, and $\log_{\mathrm{e}} 7^3$ as $3 \log_{\mathrm{e}} 7$.

2. Solving equations involving powers

Example
Solve the equation $\mathrm{e}^x = 14$.

Solution
Writing $\mathrm{e}^x = 14$ in its alternative form using logarithms we obtain $x = \log_{\mathrm{e}} 14$, which can be evaluated directly using a calculator to give 2.639.

Example
Solve the equation $e^{3x} = 14$.

Solution
Writing $e^{3x} = 14$ in its alternative form using logarithms we obtain $3x = \log_e 14 = 2.639$. Hence $x = \dfrac{2.639}{3} = 0.880$.

To solve an equation of the form $2^x = 32$ it is necessary to take the logarithm of both sides of the equation. This is referred to as 'taking logs'. Usually we use logarithms to base 10 or base e because values of these logarithms can be obtained using a scientific calculator.

Starting with $2^x = 32$, then taking logs produces $\log_{10} 2^x = \log_{10} 32$. Using the third law of logarithms, we can rewrite the left-hand side to give $x \log_{10} 2 = \log_{10} 32$. Dividing both sides by $\log_{10} 2$ gives

$$x = \frac{\log_{10} 32}{\log_{10} 2}$$

The right-hand side can now be evaluated using a calculator in order to find x:

$$x = \frac{\log_{10} 32}{\log_{10} 2} = \frac{1.5051}{0.3010} = 5$$

Hence $2^5 = 32$. Note that this answer can be checked by substitution into the original equation.

3. Solving equations involving logarithms

Example
Solve the equation $\log_{10} x = 0.98$

Solution
Rewriting the equation in its alternative form using powers gives $10^{0.98} = x$. A calculator can be used to evaluate $10^{0.98}$ to give $x = 9.550$.

Example
Solve the equation $\log_e 5x = 1.7$

Solution
Rewriting the equation in its alternative form using powers gives $e^{1.7} = 5x$. A calculator can be used to evaluate $e^{1.7}$ to give $5x = 5.4739$ so that $x = 1.095$ to 3dp.

Exercises
1. Solve each of the following equations to find x.

a) $3^x = 15$, b) $e^x = 15$, c) $3^{2x} = 9$, d) $e^{5x-1} = 17$, e) $10^{3x} = 4$.

2. Solve the equations a) $\log_e 2x = 1.36$, b) $\log_{10} 5x = 2$, c) $\log_{10}(5x + 3) = 1.2$.

Answers
1. a) 2.465, b) 2.708, c) 1, d) 0.767, e) 0.201.

2. a) 1.948, (3dp). b) 20, c) 2.570 (3dp).

