Engineering Maths First Aid Kit

2.25

Partial Fractions 3

Introduction

This leaflet describes how the partial fractions of an improper fraction can be found.

1. Partial fractions of improper fractions

An algebraic fraction is improper if the degree (highest power) of the numerator is greater than or equal to that of the denominator. Suppose we let d equal the degree of the denominator, and n the degree of the numerator. Then, in addition to the partial fractions arising from factors in the denominator we must include an additional term: this additional term is a polynomial of degree $n - d$.

Note that:

a polynomial of degree 0 is: A, a constant

a polynomial of degree 1 is: $Ax + B$

a polynomial of degree 2 is: $Ax^2 + Bx + C$,

and so on.

Example

Express $\dfrac{3x^2 + 2x}{x + 1}$ as partial fractions.

Solution

This fraction is improper because $n = 2$ and $d = 1$ and so $n \geq d$. We must include a polynomial of degree $n - d = 1$ as well as the normal partial fractions arising from the factors of the denominator. Thus

$$\frac{3x^2 + 2x}{x + 1} = Ax + B + \frac{C}{x + 1}$$

Writing the right-hand side over a common denominator gives

$$\frac{3x^2 + 2x}{x + 1} = \frac{(Ax + B)(x + 1) + C}{x + 1}$$

and so

$$3x^2 + 2x = (Ax + B)(x + 1) + C$$

As before we can equate coefficients or substitute values for x to find

$$C = 1, A = 3, \text{ and } B = -1$$

Finally

$$\frac{3x^2 + 2x}{x + 1} = 3x - 1 + \frac{1}{x + 1}$$

Example
Express $\dfrac{s^2 + 2s + 1}{s^2 + s + 1}$ in partial fractions.

Solution
Here $n = 2$, and $d = 2$. The fraction is therefore improper, with $n - d = 0$. We must include a polynomial of degree 0, that is a constant, in addition to the usual partial fractions arising from the factors of the denominator. In this example the denominator will not factorise and so this remains a quadratic factor. So,

$$\frac{s^2 + 2s + 1}{s^2 + s + 1} = A + \frac{Bs + C}{s^2 + s + 1}$$

Writing the right-hand side over a common denominator gives

$$\frac{s^2 + 2s + 1}{s^2 + s + 1} = \frac{A(s^2 + s + 1) + (Bs + C)}{s^2 + s + 1}$$

and so

$$s^2 + 2s + 1 = A(s^2 + s + 1) + (Bs + C)$$

Equating coefficients of s^2 shows that $A = 1$. Equating coefficients of s shows that $B = 1$, and you should check that $C = 0$. Hence

$$\frac{s^2 + 2s + 1}{s^2 + s + 1} = 1 + \frac{s}{s^2 + s + 1}$$

Exercises

1. Show that

$$\frac{x^4 + 2x^3 - 2x^2 + 4x - 1}{x^2 + 2x - 3} = x^2 + 1 + \frac{1}{x + 3} + \frac{1}{x - 1}$$

2. Show that

$$\frac{4x^3 + 12x^2 + 13x + 7}{4x^2 + 4x + 1} = x + 2 + \frac{2}{2x + 1} + \frac{3}{(2x + 1)^2}$$

3. Show that

$$\frac{6x^3 + x^2 + 5x - 1}{x^3 + x} = 6 - \frac{1}{x} + \frac{2x - 1}{x^2 + 1}$$

