Engineering Maths First Aid Kit

2.23

Partial Fractions 1

Introduction

An algebraic fraction can often be rewritten as the sum of simpler fractions that are called **partial fractions**. For example, it can be shown that

$$\frac{8x-12}{x^2-2x-3} \quad \text{can be written in partial fractions as} \quad \frac{3}{x-3}+\frac{5}{x+1}$$

This leaflet explains the procedure for finding partial fractions.

1. Proper and Improper fractions

When the degree of the numerator, that is the highest power on top, is less than the degree of the denominator, that is the highest power on the bottom, the fraction is said to be **proper**. The fraction

$$\frac{8x-12}{x^2-2x-3}$$

satisfies this condition and so is proper.

If a fraction is not proper it is said to be **improper**. For example, the fraction

$$\frac{2x^3+7x}{x^2+x+1}$$

is improper because the degree of the numerator, 3, is greater than the degree of the denominator, 2.

The first stage in the process of finding partial fractions is to determine whether the fraction is proper or improper because proper fractions are simpler to deal with. Improper fractions are dealt with on leaflet 2.25.

2. Finding partial fractions of proper fractions

You should carry out the following steps:

<u>Step 1</u>

Factorise the denominator if it is not already factorised.

<u>Step 2</u>

When you have factorised the denominator the factors can take various forms. You must study these forms carefully. For example you may find

$$(3x+2)(x+1)$$

These factors are both referred to as **linear factors**. Generally a linear factor has the form $ax + b$ where a and b are numbers.

The factors could be the same, as in

$$(3x + 2)(3x + 2) \qquad \text{that is} \qquad (3x + 2)^2$$

This is called a **repeated linear factor**. Generally, such a factor has the form $(ax + b)^2$.

Another possible form is

$$x^2 + x + 1$$

This is a **quadratic factor** which cannot be factorised into linear factors. Generally such a factor has the form $ax^2 + bx + c$.

It is essential that you examine the factors carefully to see which type you have. The form that the partial fractions take depends upon the type of factors obtained.

You should examine the factors of the denominator to decide which sorts of partial fraction you will need. These are summarised in the following box.

Each **linear factor**, $ax + b$, produces a partial fraction of the form

$$\frac{A}{ax + b}$$

where A represents an unknown constant which must be found.

A **repeated linear factor**, $(ax + b)^2$, produce two partial fractions of the form

$$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

where A and B represent two unknown constants which must be found.

A **quadratic factor** $ax^2 + bx + c$, which cannot be factorised, produces a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Step 3

Find the unknown constants, A, B, ... This is done using a method known as **equating coefficients**, or by substituting specific values for x, or by a mixture of both methods.

These three steps are illustrated in the examples on leaflet 2.24.